More on the Isomorphism $SU(2) \otimes SU(2) \cong SO(4)$

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Abstract

In this paper we revisit the isomorphism $SU(2) \otimes SU(2) \cong SO(4)$ to apply to some subjects in Quantum Computation and Mathematical Physics.

The unitary matrix Q by Makhlin giving the isomorphism as an adjoint action is studied and generalized from a different point of view. Some problems are also presented.

In particular, the homogeneous manifold SU(2n)/SO(2n) which characterizes entanglements in the case of n=2 is studied, and a clear-cut calculation of the universal

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1 Introduction

The purpose of this paper is to reconsider the Makhlin's theorem and to generalize it in any qubit system, and moreover to apply to some subjects in Quantum Computation and Mathematical Physics.

The isomorphism

$$SU(2) \otimes SU(2) \cong SO(4)$$

is one of well–known theorems in elementary representation theory and is a typical characteristic of four dimensional euclidean space. However, it is usually abstract, see for example [1] ¹.

In [2] Makhlin gave an interesting expression to the theorem. That is,

$$F: SU(2) \otimes SU(2) \longrightarrow SO(4), \quad F(A \otimes B) = Q^{\dagger}(A \otimes B)Q$$

with some unitary matrix $Q \in U(4)$. As far as we know this is the first that the map was given by **the adjoint action**. The construction gave and will give many applications to both Quantum Computation and Mathematical Physics, see for example [3] or [4].

In this paper we reconsider the construction (namely, Q) from a different point of view. Its construction is based on the Bell bases of 2-qubit system, so we treat a more general unitary matrix R based on them. Our method may be clear and fresh.

Next we consider the problem whether or not it is possible to construct an inclusion

$$F: SU(2) \otimes SU(2) \otimes SU(2) \longrightarrow SO(8), \quad F(A \otimes B \otimes C) = R^{\dagger}(A \otimes B \otimes C)R$$

with some unitary matrix $R \in U(8)$. A trial is made.

Since $SU(2) \otimes SU(2) \cong SO(4)$ entangled states in 2-qubit system are characterized by the homogeneous space SU(4)/SO(4) called the Lagrangean Grassmannian. In Geometry it is generalized to SU(n)/SO(n). We moreover enlarge it to U(n)/O(n), which is isomorphic to

¹This is the most standard textbook in Japan on elementary representation theory

the product space $U(1) \times SU(n)/SO(n)$. Here U(1) is a kind of phase of SU(n)/SO(n). We give an interesting coordinate system to U(n)/O(n), which is not known as far as we know.

In last we apply to so-called universal Yang-Mills action being developed by us, [5]. This is another non-linear generalization of usual Yang-Mills action [6], which is different from the Born-Infeld one [7]. To write down the action form explicitly is not so easy due to its nonlinearity. We give a clear-cut derivation to it for the abelian case.

2 The Isomorphism Revisited

In this section we review the result in [2] from a different point of view.

The 1-qubit space is $\mathbf{C}^2 = \operatorname{Vect}_{\mathbf{C}}\{|0\rangle, |1\rangle\}$ where

$$|0\rangle = \begin{pmatrix} 1\\0 \end{pmatrix}, \quad |1\rangle = \begin{pmatrix} 0\\1 \end{pmatrix}.$$
 (1)

Let $\{\sigma_1, \sigma_2, \sigma_3\}$ be the Pauli matrices :

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{2}$$

These ones act on the 1-qubit space.

Let us prepare some notations for the latter convenience. By $H(n; \mathbf{C})$ (resp. $H_0(n; \mathbf{C})$) the set of all (resp. all traceless) hermite matrices in $M(n; \mathbf{C})$

$$H(n; \mathbf{C}) = \{ A \in M(n; \mathbf{C}) \mid A^{\dagger} = A \} \supset H_0(n; \mathbf{C}) = \{ A \in H(n; \mathbf{C}) \mid \text{tr} A = 0 \}.$$

In particular, we have

$$H_0(2; \mathbf{C}) = \{ a \equiv a_1 \sigma_1 + a_2 \sigma_2 + a_3 \sigma_3 \mid a_1, a_2, a_3 \in \mathbf{R} \}.$$

Here $H(n; \mathbf{C}) \supset H(n, \mathbf{R})$ is of course the set of all real symmetric matrices.

Next, let us consider the 2–qubit space $\mathbf{C}^2\widehat{\otimes}\mathbf{C}^2\cong\mathbf{C}^4$, which is

$$\mathbf{C}^2\widehat{\otimes}\mathbf{C}^2 = \mathrm{Vect}_{\mathbf{C}}\{|00\rangle, |01\rangle, |10\rangle, |11\rangle\}$$

where $|ab\rangle = |a\rangle \otimes |b\rangle$ $(a, b \in \{0, 1\})$.

A comment is in order. We in the following use notations on tensor product which are different from usual ones ². That is,

$$\mathbf{C}^2 \otimes \mathbf{C}^2 = \{ a \otimes b \mid a, b \in \mathbf{C}^2 \},\$$

while

$$\mathbf{C}^2 \widehat{\otimes} \mathbf{C}^2 = \left\{ \sum_{j=1}^k c_j a_j \otimes b_j \mid a_j, b_j \in \mathbf{C}^2, \ c_j \in \mathbf{C}, \ k \in \mathbf{N} \right\} \cong \mathbf{C}^4.$$

We consider the Bell bases $\{|\Psi_1\rangle, |\Psi_2\rangle, |\Psi_3\rangle, |\Psi_4\rangle\}$ which are defined as

$$|\Psi_{1}\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle), \quad |\Psi_{2}\rangle = \frac{1}{\sqrt{2}}(|01\rangle + |10\rangle),$$

$$|\Psi_{3}\rangle = \frac{1}{\sqrt{2}}(|01\rangle - |10\rangle), \quad |\Psi_{4}\rangle = \frac{1}{\sqrt{2}}(|00\rangle - |11\rangle). \tag{3}$$

By making use of the bases the unitary matrix R is defined as

$$R = \left(e^{i\theta_1} |\Psi_1\rangle, e^{i\theta_2} |\Psi_2\rangle, e^{i\theta_3} |\Psi_3\rangle, e^{i\theta_4} |\Psi_4\rangle \right) \in U(4).$$

In the matrix form

$$R = \frac{1}{\sqrt{2}} \begin{pmatrix} e^{i\theta_1} & 0 & 0 & e^{i\theta_4} \\ 0 & e^{i\theta_2} & e^{i\theta_3} & 0 \\ 0 & e^{i\theta_2} & -e^{i\theta_3} & 0 \\ e^{i\theta_1} & 0 & 0 & -e^{i\theta_4} \end{pmatrix}. \tag{4}$$

It is well–known the isomorphism

$$F: SU(2) \otimes SU(2) \cong SO(4).$$

To realize it as an adjoint action by R (if it is possible)

$$F(A \otimes B) = R^{\dagger}(A \otimes B)R \in SO(4), \tag{5}$$

we have only to determine $\{e^{i\theta_1}, e^{i\theta_2}, e^{i\theta_3}, e^{i\theta_4}\}$ the coefficients of R. Let us consider this problem in a Lie algebra level because it is in general not easy to treat it in a Lie group level.

²We believe that our notations are clearer than usual ones

Since the Lie algebra of $SU(2) \otimes SU(2)$ is

$$\mathfrak{L}(SU(2)\otimes SU(2)) = \{i(a\otimes 1_2 + 1_2\otimes b) \mid a,b\in H_0(2;\mathbf{C})\},\,$$

we have only to examine

$$f(i(a \otimes 1_2 + 1_2 \otimes b)) = iR^{\dagger}(a \otimes 1_2 + 1_2 \otimes b)R \in \mathfrak{L}(SO(4)). \tag{6}$$

By setting $a = \sum_{j=1}^{3} a_j \sigma_j$ and $b = \sum_{j=1}^{3} b_j \sigma_j$ let us calculate the right hand side of (6). The result is

$$iR^{\dagger}(a \otimes 1_{2} + 1_{2} \otimes b)R =$$

$$\begin{pmatrix} 0 & ie^{-i(\theta_{1} - \theta_{2})}(a_{1} + b_{1}) & -e^{-i(\theta_{1} - \theta_{3})}(a_{2} - b_{2}) & ie^{-i(\theta_{1} - \theta_{4})}(a_{3} + b_{3}) \\ ie^{i(\theta_{1} - \theta_{2})}(a_{1} + b_{1}) & 0 & ie^{-i(\theta_{2} - \theta_{3})}(a_{3} - b_{3}) & -e^{-i(\theta_{2} - \theta_{4})}(a_{2} + b_{2}) \\ e^{i(\theta_{1} - \theta_{3})}(a_{2} - b_{2}) & ie^{i(\theta_{2} - \theta_{3})}(a_{3} - b_{3}) & 0 & -ie^{-i(\theta_{3} - \theta_{4})}(a_{1} - b_{1}) \\ ie^{i(\theta_{1} - \theta_{4})}(a_{3} + b_{3}) & e^{i(\theta_{2} - \theta_{4})}(a_{2} + b_{2}) & -ie^{i(\theta_{3} - \theta_{4})}(a_{1} - b_{1}) & 0 \end{pmatrix} .$$
 (7)

Here if we set

$$ie^{-i(\theta_1-\theta_2)} = 1$$
, $ie^{-i(\theta_1-\theta_4)} = 1$, $ie^{-i(\theta_2-\theta_3)} = 1$, $-ie^{-i(\theta_3-\theta_4)} = 1$,

from which $-e^{-i(\theta_1-\theta_3)}=1$ and $-e^{-i(\theta_2-\theta_4)}=-1$ automatically, then we have

$$e^{i\theta_1} = 1$$
, $e^{i\theta_2} = -i$, $e^{i\theta_3} = -1$, $e^{i\theta_4} = -i$.

Therefore our R becomes

$$R = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 & 0 & -i \\ 0 & -i & -1 & 0 \\ 0 & -i & 1 & 0 \\ 1 & 0 & 0 & i \end{pmatrix}.$$
 (8)

We used the notation R again for simplicity. Note that the unitary matrix R is a bit different from Q in [2].

For the latter convenience let us rewrite. If we set

$$iR^{\dagger}(a \otimes 1_{2} + 1_{2} \otimes b)R = \begin{pmatrix} 0 & a_{1} + b_{1} & a_{2} - b_{2} & a_{3} + b_{3} \\ -(a_{1} + b_{1}) & 0 & a_{3} - b_{3} & -(a_{2} + b_{2}) \\ -(a_{2} - b_{2}) & -(a_{3} - b_{3}) & 0 & a_{1} - b_{1} \\ -(a_{3} + b_{3}) & a_{2} + b_{2} & -(a_{1} - b_{1}) & 0 \end{pmatrix}$$

$$\equiv \begin{pmatrix} 0 & f_{12} & f_{13} & f_{14} \\ -f_{12} & 0 & f_{23} & f_{24} \\ -f_{13} & -f_{23} & 0 & f_{34} \\ -f_{14} & -f_{24} & -f_{34} & 0 \end{pmatrix} \in \mathfrak{L}(SO(4))$$

$$(9)$$

then we obtain

$$a = a_1 \sigma_1 + a_2 \sigma_2 + a_3 \sigma_3 = \frac{f_{12} + f_{34}}{2} \sigma_1 + \frac{f_{13} - f_{24}}{2} \sigma_2 + \frac{f_{14} + f_{23}}{2} \sigma_3, \tag{10}$$

$$b = b_1 \sigma_1 + b_2 \sigma_2 + b_3 \sigma_3 = \frac{f_{12} - f_{34}}{2} \sigma_1 - \frac{f_{13} + f_{24}}{2} \sigma_2 + \frac{f_{14} - f_{23}}{2} \sigma_3.$$
 (11)

3 A Trial toward Generalization

We would like to generalize the result $SU(2) \otimes SU(2) \cong SO(4)$ in the preceding section. Of course it is not true that $SU(2) \otimes SU(2) \otimes SU(2) \cong SO(8)$. Our question is as follows: is it possible to find an inclusion

$$F: SU(2) \otimes SU(2) \otimes SU(2) \longrightarrow SO(8)$$

with the form

$$F(A \otimes B \otimes C) = R^{\dagger}(A \otimes B \otimes C)R \in SO(8)$$
(12)

by finding a unitary matrix $R \in U(8)$?

Let us make a trial in the following. In the preceding section we used the Bell bases to construct the unitary matrix R, so in this case we trace the same line 3 .

³We believe the way of thinking natural

In 3–qudit system the generalized "Bell bases" are known to be $\{|\Psi_1\rangle, |\Psi_2\rangle, |\Psi_3\rangle, |\Psi_4\rangle, |\Psi_5\rangle, |\Psi_6\rangle, |\Psi_7\rangle, |\Psi_8\rangle\} \text{ where}$

$$|\Psi_{1}\rangle = \frac{1}{\sqrt{2}}(|000\rangle + |111\rangle), \quad |\Psi_{2}\rangle = \frac{1}{\sqrt{2}}(|001\rangle + |110\rangle),$$

$$|\Psi_{3}\rangle = \frac{1}{\sqrt{2}}(|010\rangle + |101\rangle), \quad |\Psi_{4}\rangle = \frac{1}{\sqrt{2}}(|011\rangle + |100\rangle),$$

$$|\Psi_{5}\rangle = \frac{1}{\sqrt{2}}(|011\rangle - |100\rangle), \quad |\Psi_{6}\rangle = \frac{1}{\sqrt{2}}(|010\rangle - |101\rangle),$$

$$|\Psi_{7}\rangle = \frac{1}{\sqrt{2}}(|001\rangle - |110\rangle), \quad |\Psi_{8}\rangle = \frac{1}{\sqrt{2}}(|000\rangle - |111\rangle),$$
(13)

see for example [8]. Then the unitary matrix R corresponding to (4) is given by

$$R = \left(e^{i\theta_1} | \Psi_1 \rangle, e^{i\theta_2} | \Psi_2 \rangle, e^{i\theta_3} | \Psi_3 \rangle, e^{i\theta_4} | \Psi_4 \rangle, e^{i\theta_5} | \Psi_5 \rangle, e^{i\theta_6} | \Psi_6 \rangle, e^{i\theta_7} | \Psi_7 \rangle, e^{i\theta_8} | \Psi_8 \rangle \right) \in U(8),$$

or in the matrix form

$$R = \frac{1}{\sqrt{2}} \begin{pmatrix} e^{i\theta_1} & 0 & 0 & 0 & 0 & 0 & e^{i\theta_8} \\ 0 & e^{i\theta_2} & 0 & 0 & 0 & 0 & e^{i\theta_7} & 0 \\ 0 & 0 & e^{i\theta_3} & 0 & 0 & e^{i\theta_6} & 0 & 0 \\ 0 & 0 & 0 & e^{i\theta_4} & e^{i\theta_5} & 0 & 0 & 0 \\ 0 & 0 & 0 & e^{i\theta_4} & -e^{i\theta_5} & 0 & 0 & 0 \\ 0 & 0 & e^{i\theta_3} & 0 & 0 & -e^{i\theta_6} & 0 & 0 \\ 0 & e^{i\theta_2} & 0 & 0 & 0 & 0 & -e^{i\theta_7} & 0 \\ e^{i\theta_1} & 0 & 0 & 0 & 0 & 0 & -e^{i\theta_8} \end{pmatrix}.$$
(14)

We must check whether or not it is possible to construct

$$F(A \otimes B \otimes C) = R^{\dagger}(A \otimes B \otimes C)R \in SO(8) \tag{15}$$

by determining the coefficients $\{e^{i\theta_1}, e^{i\theta_2}, e^{i\theta_3}, e^{i\theta_4}, e^{i\theta_5}, e^{i\theta_6}, e^{i\theta_7}, e^{i\theta_8}\}.$

Similarly in the preceding section we have only to check it in a Lie algebra level:

$$f(i(a\otimes 1_2\otimes 1_2+1_2\otimes b\otimes 1_2+1_2\otimes 1_2\otimes c))=iR^{\dagger}(a\otimes 1_2\otimes 1_2+1_2\otimes b\otimes 1_2+1_2\otimes 1_2\otimes c)R\in \mathfrak{L}(SO(8)).$$

The result is negative. That is, the coefficients $\{e^{i\theta_1}, e^{i\theta_2}, e^{i\theta_3}, e^{i\theta_4}, e^{i\theta_5}, e^{i\theta_6}, e^{i\theta_7}, e^{i\theta_8}\}$ satisfying the above equation don't exist (a long calculation like (7) is omitted). Therefore we again propose

Problem Does a unitary matrix $R \in SU(8)$ exist giving an inclusion

$$F: SU(2) \otimes SU(2) \otimes SU(2) \longrightarrow SO(8), \quad F(A \otimes B \otimes C) = R^{\dagger}(A \otimes B \otimes C)R$$
?

A comment is in order. If we can find such an inclusion then we have the following fiber bundle

$$\frac{SO(8)}{F(SU(2)\otimes SU(2)\otimes SU(2))} \longrightarrow \frac{SU(8)}{F(SU(2)\otimes SU(2)\otimes SU(2))} \longrightarrow \frac{SU(8)}{SO(8)}.$$

That is, entangled states for 3-qubit system are characterized by the homogeneous space $SU(8)/F(SU(2)\otimes SU(2)\otimes SU(2))$ and this space is understood by the fiber bundle.

4 General R Matrix

In this section we treat the general n-qubit system. We would like to generalize the unitary matrix R in the preceding sections. Generalized "Bell bases" are constructed as follows.

For $0 \le k \le 2^{n-1} - 1$, since k can be written as

$$k = a_0 2^{n-2} + a_1 2^{n-3} + \dots + a_{n-3} 2 + a_{n-2}, \quad a_j \in \{0, 1\}$$

we set

$$|\Psi_{k+1}\rangle = \frac{1}{\sqrt{2}} \left\{ |0a_0a_1 \cdots a_{n-2}\rangle + |1\check{a}_0\check{a}_1 \cdots \check{a}_{n-2}\rangle \right\},$$

$$|\Psi_{2^n-k}\rangle = \frac{1}{\sqrt{2}} \left\{ |0a_0a_1 \cdots a_{n-2}\rangle - |1\check{a}_0\check{a}_1 \cdots \check{a}_{n-2}\rangle \right\}$$
(16)

where $\ddot{a}_j = 1 - a_j$.

We define the unitary matrix R as

$$R = \left(e^{i\theta_1}|\Psi_1\rangle, e^{i\theta_2}|\Psi_2\rangle, \cdots, e^{i\theta_{2^{n-1}}}|\Psi_{2^{n-1}}\rangle, e^{i\theta_{2^{n-1}+1}}|\Psi_{2^{n-1}+1}\rangle, \cdots, e^{i\theta_{2^{n-1}}}|\Psi_{2^{n-1}}\rangle, e^{i\theta_{2^n}}|\Psi_{2^n}\rangle\right).$$

In the matrix form

$$R = \frac{1}{\sqrt{2}} \begin{pmatrix} e^{i\theta_{1}} & e^{i\theta_{2}n} \\ e^{i\theta_{2}} & e^{i\theta_{2}n-1} \\ & \ddots & & \ddots \\ & e^{i\theta_{2}n-1} & e^{i\theta_{2}n-1+1} \\ & & e^{i\theta_{2}n-1} & -e^{i\theta_{2}n-1+1} \\ & & \ddots & & \ddots \\ & & & & \ddots \\ & & & & & -e^{i\theta_{2}n} \end{pmatrix} . \tag{17}$$

For n=2 and 3 the matrix R in the preceding sections is recovered.

Now we give a characterization to R. That is, R satisfies the equation

$$R(\sigma_3 \otimes 1_2 \otimes \cdots \otimes 1_2)R^{\dagger} = \sigma_1 \otimes \sigma_1 \otimes \cdots \otimes \sigma_1. \tag{18}$$

The proof is left to readers (check this for (4) and (14)).

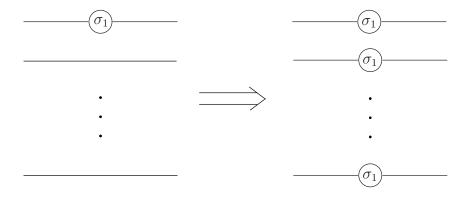
If we define

$$\widetilde{W} = W \otimes 1_2 \otimes \cdots \otimes 1_2$$

where W is the usual Walsh–Hadamard matrix $(W\sigma_3W=\sigma_1)$, then the product \widetilde{RW} gives

$$R\widetilde{W}(\sigma_1 \otimes 1_2 \otimes \cdots \otimes 1_2)(R\widetilde{W})^{\dagger} = \sigma_1 \otimes \sigma_1 \otimes \cdots \otimes \sigma_1. \tag{19}$$

That is, \widetilde{RW} is a copy operation for σ_1 :



We believe that R will play an important role in Quantum Computation.

5 Application to Elementary Representation Theory

To construct a representation from SU(2) to SO(3) is very well-known. In this section we point out a relation between the representation ρ and the isomorphism F in the section 2.

First of all let us make a review of constructing the representation,

$$\rho: SU(2) \longrightarrow SO(3), \tag{20}$$

see for example [9], Appendix F. For the matrix

$$g = \begin{pmatrix} a+ib & c+id \\ -c+id & a-ib \end{pmatrix} \quad a,b,c,d \in \mathbf{R}$$

it is easy to see

$$g \in SU(2) \iff a^2 + b^2 + c^2 + d^2 = 1.$$

For the Pauli matrices $\{\sigma_1, \sigma_2, \sigma_3\}$ in (2) we set

$$\tau_j = \frac{1}{2}\sigma_j$$
 for $j = 1, 2, 3$.

Then the representation ρ is constructed as follows: since

$$g\tau_1g^{-1} = (a^2 - b^2 - c^2 + d^2)\tau_1 - 2(ab - cd)\tau_2 + 2(ac + bd)\tau_3,$$

$$g\tau_2g^{-1} = 2(ab + cd)\tau_1 + (a^2 - b^2 + c^2 - d^2)\tau_2 - 2(ad - bc)\tau_3,$$

$$g\tau_3g^{-1} = -2(ac - bd)\tau_1 + 2(ad + bc)\tau_2 + (a^2 + b^2 - c^2 - d^2)\tau_3,$$

we have

$$(g\tau_1g^{-1}, g\tau_2g^{-1}, g\tau_3g^{-1}) = (\tau_1, \tau_2, \tau_3) \rho(g)$$

where

$$\rho(g) = \begin{pmatrix}
a^2 - b^2 - c^2 + d^2 & 2(ab + cd) & -2(ac - bd) \\
-2(ab - cd) & a^2 - b^2 + c^2 - d^2 & 2(ad + bc) \\
2(ac + bd) & -2(ad - bc) & a^2 + b^2 - c^2 - d^2
\end{pmatrix}.$$
(21)

Here let us rewrite the above. Noting that

$$a^{2} - b^{2} - c^{2} + d^{2} = 1 - 2(b^{2} + c^{2}), \quad a^{2} - b^{2} + c^{2} - d^{2} = 1 - 2(b^{2} + d^{2}),$$

 $a^{2} + b^{2} - c^{2} - d^{2} = 1 - 2(c^{2} + d^{2}),$

from $a^2 + b^2 + c^2 + d^2 = 1$, we obtain

$$\rho(g) = \begin{pmatrix}
1 - 2(b^2 + c^2) & 2(ab + cd) & -2(ac - bd) \\
-2(ab - cd) & 1 - 2(b^2 + d^2) & 2(ad + bc) \\
2(ac + bd) & -2(ad - bc) & 1 - 2(c^2 + d^2)
\end{pmatrix}.$$
(22)

Now we would like to search a relation induced from ρ and F. See the following diagram:

where ι is the natural inclusion defined by

$$\iota(O) = \begin{pmatrix} 1 \\ O \end{pmatrix} \in SO(4) \text{ for } O \in SO(3).$$

We determine the map $\tilde{\iota}$ defined by

$$\tilde{\iota} = F^{-1} \circ \iota \circ \rho \implies \tilde{\iota}(g) = R \begin{pmatrix} 1 \\ \rho(g) \end{pmatrix} R^{\dagger} \text{ for } g \in SU(2).$$

The result is

$$\tilde{\iota}\left(\begin{pmatrix} a+ib & c+id \\ -c+id & a-ib \end{pmatrix}\right) = \begin{pmatrix} a+ib & c+id \\ -c+id & a-ib \end{pmatrix} \otimes \begin{pmatrix} a-ib & c-id \\ -c-id & a+ib \end{pmatrix}.$$
(23)

Let us rewrite the equation. It is easy to see

The right hand side of (23) =
$$g \otimes \sigma_2 g \sigma_2^{\dagger} = (1_2 \otimes \sigma_2)(g \otimes g)(1_2 \otimes \sigma_2)^{\dagger}$$
,

so we have

$$\tilde{\iota}(g) = (1_2 \otimes \sigma_2)(g \otimes g)(1_2 \otimes \sigma_2)^{\dagger} \equiv (1_2 \otimes \sigma_2)\Delta(g)(1_2 \otimes \sigma_2)^{\dagger}. \tag{24}$$

A comment is in order. If we rewrite (24) as

$$\begin{pmatrix} 1 \\ \rho(g) \end{pmatrix} \equiv 1 \oplus \rho(g) = R^{\dagger}(1_2 \otimes \sigma_2)(g \otimes g) \left\{ R^{\dagger}(1_2 \otimes \sigma_2) \right\}^{\dagger},$$

this is the well–known irreducible decomposition of tensor product $\frac{1}{2} \otimes \frac{1}{2} = 0 \oplus 1$ in terms of spin representation, compare this with [10].

6 Lagrangean Grassmannians

Since $SU(2) \otimes SU(2) \cong SO(4)$, entangled states in 2-qubit system are characterized by the homogeneous space SU(4)/SO(4), which is a special case of the Lagrangean Grassmannians SU(n)/SO(n) for $n \geq 2$. Here let us expand SU(n)/SO(n) a little bit, namely our target is U(n)/O(n). See for example [11] for the Grassmannians.

For $n \geq 1$ we define the two sets (spaces)

$$X_n \equiv \left\{ R \in U(n) \mid R^T = R \right\},\tag{25}$$

$$\widetilde{X}_n \equiv \left\{ R \in SU(n) \mid R^T = R \right\}. \tag{26}$$

Then it is easy to see

$$X_n = \left\{ A^T A \mid A \in U(n) \right\} \cong \frac{U(n)}{O(n)}, \tag{27}$$

$$\widetilde{X}_n = \left\{ A^T A \mid A \in SU(n) \right\} \cong \frac{SU(n)}{SO(n)}$$
(28)

and

$$X_n \cong U(1) \times \widetilde{X}_n \tag{29}$$

because $R \in X_n$ can be written as $R = e^{i\theta} \tilde{R}$ where $\tilde{R} \in \tilde{X}_n$.

We can give a very interesting coordinate system (as a manifold) to X_n ⁴.

Let $X_n \ni R_0 = A^T A$ be fixed and define a neighborhood of R_0 as

$$U_A = \{ R \in X_n \mid R + R_0 \in GL(n; \mathbf{C}) \}$$

and define a map

$$\phi_A: U_A \longrightarrow H(n; \mathbf{R}), \quad \phi_A(R) = 2i \left(E + \bar{A}R\bar{A}^T\right)^{-1} \left(E - \bar{A}R\bar{A}^T\right).$$
 (30)

⁴However, it is not easy for \widetilde{X}_n

It is easy to see

$$\phi_A(R) = 2i \left\{ 2 \left(E + \bar{A}R\bar{A}^T \right)^{-1} - E \right\}.$$

Conversely, we define a map

$$\omega_A: H(n; \mathbf{R}) \longrightarrow U_A, \quad \omega_A(X) = A^T \left(E + \frac{i}{2} X \right) \left(E - \frac{i}{2} X \right)^{-1} A.$$
 (31)

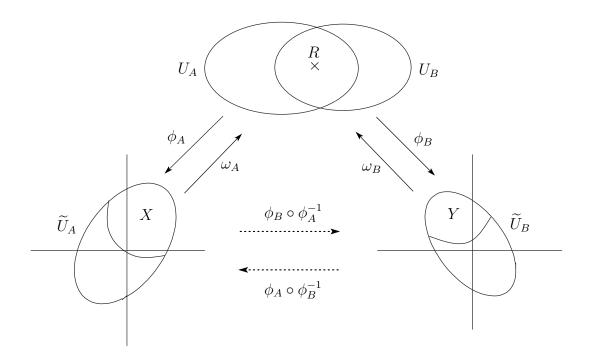
Then it is not difficult to see

$$\omega_A \circ \phi_A = 1_{U_A}$$
 and $\phi_A \circ \omega_A = 1_{H(n;\mathbf{R})}$.

Next, for $U_A \cap U_B \ni R$ satisfying $R = \omega_A(X) = \omega_B(Y)$ it is not difficult to solve

$$Y = 2\left(\alpha - \frac{1}{2}X\beta\right)^{-1}\left(\beta + \frac{1}{2}X\alpha\right) \iff Y = \phi_B \circ \phi_A^{-1}(X)$$
 (32)

if we set $AB^\dagger=\alpha+i\beta.$ See the following diagram :



The aim of the paper is not to study detailed (geometric) structures of $X_n \cong U(n)/O(n)$, for the case of n=4 especially, so we leave it in a forthcoming paper.

7 Application to Universal Yang-Mills Action

In this section we revisit [5] from a different point of view. In the following we consider the \mathbb{C}^{∞} category, namely \mathbb{C}^{∞} -manifolds, \mathbb{C}^{∞} -maps, etc. See [12] for more detailed descriptions.

Let M be a four dimensional manifold and G a classical group, in particular, U(1) and SU(2). We denote by $\{G, P, \pi, M\}$ a principal G bundle on M

$$\pi: P \longrightarrow M, \quad \pi^{-1}(m) \cong G.$$

We consider a theory of connections of the principal G bundle, so let $g \equiv \mathfrak{L}(G)$ be the Lie algebra of the group G. In the following we treat it locally, which is enough for our purpose. Let U be any open set in M, then a connection $\{A_{\mu}\}$ (a gauge potential) is

$$A_{\mu}: U \longrightarrow g$$

and the corresponding curvature $\{F_{\mu\nu}\}$ is given by

$$F_{\mu\nu} = \frac{\partial A_{\nu}}{\partial x_{\mu}} - \frac{\partial A_{\mu}}{\partial x_{\nu}} + [A_{\mu}, A_{\nu}] \equiv \partial_{\mu} A_{\nu} - \partial_{\nu} A_{\mu} + [A_{\mu}, A_{\nu}].$$

We note that

$$F_{\mu\nu}: U \longrightarrow g, \qquad F_{\nu\mu} = -F_{\mu\nu}.$$

For a map $\phi: U \longrightarrow G$, a gauge transformation by ϕ is defined by

$$A_{\mu} \longrightarrow \phi^{-1} A_{\mu} \phi + \phi^{-1} \partial_{\mu} \phi$$

and the curvature is then transformed like

$$F_{\mu\nu} \longrightarrow \phi^{-1}F_{\mu\nu}\phi.$$

In the following we consider only the abelian case G=U(1). For that let us define a curvature matrix

$$\mathcal{F} = \begin{pmatrix} 0 & F_{12} & F_{13} & F_{14} \\ -F_{12} & 0 & F_{23} & F_{24} \\ -F_{13} & -F_{23} & 0 & F_{34} \\ -F_{14} & -F_{24} & -F_{34} & 0 \end{pmatrix}. \tag{33}$$

The abelian Yang–Mills action \mathcal{A}_{YM} ⁵ is given by

$$\mathcal{A}_{YM} \equiv \frac{1}{2} \operatorname{tr} (g\mathcal{F})^2 = -g^2 \sum_{i \le j} F_{ij}^2$$
(34)

where g is a coupling constant, see [6].

This model is very well–known and there is nothing added furthermore. We are interested in some non–linear generalization(s) of it.

A non–linear extension of the abelian Yang–Mills is known as the Born–Infeld theory whose action \mathcal{A}_{BI} is given by

$$A_{BI} \equiv \sqrt{\det\left(\mathbf{1}_4 + g\mathcal{F}\right)} \tag{35}$$

where g is a coupling constant, see [7].

As to this model and its generalizations to non–abelian groups there are a lot of papers. However, we don't make a comment in the paper.

On the other hand, we have presented another non-linear extension, [5]. Its action \mathcal{A} is given by

$$\mathcal{A}_{FOS} \equiv \text{tr } e^{g\mathcal{F}} \tag{36}$$

where g is a coupling constant. We have called this the universal Yang–Mills action.

Let us calculate the right hand side of (36) by making use of the result in section 2, which is more clear—cut. From (9) with (10) and (11)

$$e^{g\mathcal{F}} = R^{\dagger} R e^{g\mathcal{F}} R^{\dagger} R = R^{\dagger} e^{gR\mathcal{F}R^{\dagger}} R,$$

while

$$R\mathcal{F}R^{\dagger} = i(a \otimes 1_2 + 1_2 \otimes b)$$

with

$$a = \frac{F_{12} + F_{34}}{2}\sigma_1 + \frac{F_{13} - F_{24}}{2}\sigma_2 + \frac{F_{14} + F_{23}}{2}\sigma_3, \tag{37}$$

$$b = \frac{F_{12} - F_{34}}{2}\sigma_1 - \frac{F_{13} + F_{24}}{2}\sigma_2 + \frac{F_{14} - F_{23}}{2}\sigma_3.$$
 (38)

⁵ it may be suitable to call it the Maxwell action

Therefore

$$e^{gR\mathcal{F}R^{\dagger}} = e^{ig(a\otimes 1_2 + 1_2\otimes b)} = e^{iga} \otimes e^{igb}$$

and

$$\operatorname{tr} e^{g\mathcal{F}} = \operatorname{tr} e^{gR\mathcal{F}R^{\dagger}} = \operatorname{tr} e^{iga} \otimes e^{igb} = \operatorname{tr} e^{iga} \operatorname{tr} e^{igb}.$$

Noting the well–known formula

$$e^{i(x\sigma_1 + y\sigma_2 + z\sigma_3)} = \cos r 1_2 + \frac{\sin r}{r} i(x\sigma_1 + y\sigma_2 + z\sigma_3), \quad r \equiv \sqrt{x^2 + y^2 + z^2}$$

we finally obtain

$$\mathcal{A}_{FOS} \equiv \text{tr } e^{g\mathcal{F}} = 4\cos(gX_{asd})\cos(gX_{sd})$$
(39)

where

$$X_{sd}^{2} = \frac{1}{4} \left\{ (F_{12} - F_{34})^{2} + (F_{13} + F_{24})^{2} + (F_{14} - F_{23})^{2} \right\}, \tag{40}$$

$$X_{asd}^{2} = \frac{1}{4} \left\{ (F_{12} + F_{34})^{2} + (F_{13} - F_{24})^{2} + (F_{14} + F_{23})^{2} \right\}. \tag{41}$$

It is very notable that

$$X_{sd} = 0 \iff F_{12} = F_{34}, \ F_{13} = -F_{24}, \ F_{14} = F_{23} \iff \{F_{ij}\} \text{ is self-dual},$$

$$X_{asd} = 0 \iff F_{12} = -F_{34}, \ F_{13} = F_{24}, \ F_{14} = -F_{23} \iff \{F_{ij}\} \text{ is anti-self-dual.}$$

The characteristic of our model is that the action (39) splits **automatically** into two parts consisting of self–dual and anti–self–dual directions. Namely, we have **automatically** the self–dual and anti–self–dual equations without solving the equations of motion as in a usual case.

Last in this section let us make an important comment on the action (39). Since the target (as a map) of curvatures $\{F_{ij}\}$ is the abelian Lie algebra $u(1) = \sqrt{-1}\mathbf{R}$ it may be appropriate to write $F_{ij} = \sqrt{-1}G_{ij}$. Then (39) is changed into the more suitable form

$$\mathcal{A}_{FOS} = 4 \cosh(gY_{asd}) \cosh(gY_{sd}) \tag{42}$$

with

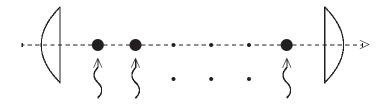
$$Y_{sd}^{2} = \frac{1}{4} \left\{ (G_{12} - G_{34})^{2} + (G_{13} + G_{24})^{2} + (G_{14} - G_{23})^{2} \right\} \ge 0, \tag{43}$$

$$Y_{asd}^2 = \frac{1}{4} \left\{ (G_{12} + G_{34})^2 + (G_{13} - G_{24})^2 + (G_{14} + G_{23})^2 \right\} \ge 0.$$
 (44)

8 Discussion

In this letter we studied several aspects coming from the isomorphism $SU(2) \otimes SU(2) \cong SO(4)$ in Quantum Computation and Mathematical Physics and moreover presented the problem concerning a possibility of generalization. Details of the paper and further developments will be published elsewhere.

In last let us make a brief comment. We are studying a quantum computation based on Cavity QED and have presented a model, see [13] and [14] in detail. The image is as follows (the details are omitted):



The general setting for a quantum computation based on Cavity QED:

the dotted line means a single photon inserted in the cavity and
all curves mean external laser fields subjected to atoms

It seems to us that applying some results in the paper to the model is not difficult. This is also our forthcoming target.

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